Necessary optimality conditions in one control problem with a multipoint quality functional

V.G. Rzayeva

Sumgait State University, Sumgait, Azerbaijan

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ABSTRACT

We consider the problem of finding the minimum value of a multipoint functional quality determined on solutions by systems of nonlinear equations of the Volterra type. Taking one version of the increment method, the necessary optimality conditions are proved.

1. Introduction

Various necessary optimality conditions such as Pontryagin's maximum principle in various optimal control problems described by ordinary differential equations are proved in [1-4] and others. The studies [5-9] and others are devoted to the qualitative theory of optimal control of Goursat-Darboux systems.

Many processes are also described by various integro-differential and integral equations (see e.g., [10-14]).

In this paper, we consider one optimal control problem described by a system of two-dimensional integral equations of the Volterra type. Analogs of L.S. Pontryagin's maximum principle [1-3] and the linearized maximum condition [2-4]. In the case of the open control domain, an analogue of the Euler equation is established.

2. Problem statement

Suppose $D = [t_0, t_1] \times [x_0, x_1]$ is a given rectangle, and $(T_i, X_i), i = \overline{1,k}, (t_0 < T_1 < \cdots < T_k < t_1, x_0 < X_1 < \cdots < X_k < x_1)$ are given points.

Let it be required to find the minimum value of the multipoint functional

$$S(u) = \varphi\big(z(T_1, X_1), z(T_2, X_2), \dots, z(T_k, X_k)\big), \tag{1}$$

with the constraints

E-mail address: kmansimov@mail.ru (V.G. Rzayeva).

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$$u(t,x) \in U \subset R^r, (t,x) \in D, \tag{2}$$

$$z(t,x) = \int_{t_0}^{t} \int_{x_0}^{x} f\big(t,x,\tau,s,z(\tau,s),u(\tau,s)\big) ds d\tau. \tag{3}$$

Here, $\varphi(a_1, \dots, a_k)$ is a given continuously differentiable scalar function, U is a given, non-empty, bounded set, $u(t,x)$ is a r-dimensional measurable and bounded control function, and $f(t,x,\tau,s,z,u)$ is a given n-dimensional vector-function, continuous in the set of variables together with partial derivatives with respect to z.

Each control function with the above properties will be called an admissible control.

It is assumed that for each given admissible control $u(t,x)$, integral equation (3) has a unique continuous solution.

The admissible control $\big(u(t,x), z(t,x)\big)$, which is the solution to the set problem, will be called an optimal process.

Let us proceed to the derivation of the necessary optimality conditions in the problem under investigation.

3. An analogue of L.S. Pontryagin's maximum principle

Suppose $\big(u(t,x), z(t,x)\big)$ is a fixed, and $\big(\bar{u}(t,x), \bar{z}(t,x)\big)$ arbitrary admissible processes.

If we introduce the notation $\Delta u(t,x) = \bar{u}(t,x) - u(t,x), \Delta z(t,x) = \bar{z}(t,x) - z(t,x)$, we get from (3) that the increment $\Delta z(t,x)$ of the state $z(t,x)$ is a solution to the integral equation

$$\Delta z(t,x) = \int_{t_0}^{t} \int_{x_0}^{x} \big[f\big(t,x,\tau,s,\bar{z}(\tau,s),\bar{u}(\tau,s)\big) - f\big(t,x,\tau,s,z(\tau,s),u(\tau,s)\big)\big] ds d\tau. \tag{4}$$

Suppose $\psi(t,x)$ is an as yet arbitrary n-dimensional vector-function.

Multiplying both sides of relation (4) on the left scalarwise by $\psi(t,x)$, and then integrating both sides of the resulting relation with respect to D, we get

$$\int_{t_0}^{t_1} \int_{x_0}^{x_1} \psi'(t,x) \Delta z(t,x) dx dt =$$

$$= \int_{t_0}^{t_1} \int_{x_0}^{x_1} \psi'(t,x) \left[\int_{t_0}^{t} \int_{x_0}^{x} \big[f\big(t,x,\tau,s,\bar{z}(\tau,s),\bar{u}(\tau,s)\big) - \right.$$

$$\left. - f\big(t,x,\tau,s,z(\tau,s),u(\tau,s)\big)\big] ds d\tau \right] dx dt. \tag{5}$$

Here and in what follows, the prime ($'$) means a transposition operation.

Applying the Fubini formula (see, e.g., [6]) to the right-hand side of relation (5), we will have

$$\int_{t_0}^{t_1} \int_{x_0}^{x_1} \psi'(t,x) \Delta z(t,x) dx dt =$$

$$\int_{t_0}^{t}\int_{x_0}^{x}\left[\int_{t}^{t_1}\int_{x}^{x_1}\psi'(\tau,s)\big[f\big(\tau,s,t,x,\bar{z}(t,x),\bar{u}(t,x)\big)-f\big(\tau,s,t,x,z(t,x),u(t,x)\big)\big]\,dxdt\right]dsd\tau \quad (6)$$

Identity (6) allows us to write down the increment of functional (1) corresponding to the admissible controls $u(t,x)$ and $\bar{u}(t,x)$ in the form

$$\Delta S(u)=S(\bar{u})-S(u)=\varphi\big(\bar{z}(T_1,X_1),\bar{z}(T_2,X_2),\dots,\bar{z}(T_k,X_k)\big)-$$

$$-\varphi\big(z(T_1,X_1),z(T_2,X_2),\dots,z(T_k,X_k)\big)+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\psi'(t,x)\Delta z(t,x)dxdt-$$

$$-\int_{t_0}^{t}\int_{x_0}^{x}\left[\int_{t}^{t_1}\int_{x}^{x_1}\psi'(\tau,s)\big[f\big(\tau,s,t,x,\bar{z}(t,x),\bar{u}(t,x)\big)-f\big(\tau,s,t,x,z(t,x),u(t,x)\big)\big]dxdt\right]dsd\tau \quad (7)$$

Using the Taylor formula, we obtain the expansions

$$\varphi\big(\bar{z}(T_1,X_1),\bar{z}(T_2,X_2),\dots,\bar{z}(T_k,X_k)\big)-\varphi\big(z(T_1,X_1),z(T_2,X_2),\dots,z(T_k,X_k)\big)=$$

$$=\sum_{i=1}^{k}\frac{\partial\varphi'\big(z(T_1,X_1),z(T_2,X_2),\dots,z(T_k,X_k)\big)}{\partial a_i}\Delta z(T_i,X_i)+o_1\left(\sum_{i=1}^{k}\|\Delta z(T_i,X_i)\|\right), \quad (8)$$

where $\|\alpha\|$ is the norm of the vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)'$ determined from the formula

$$\|\alpha\|=\sum_{i=1}^{k}\|\alpha_i\|,$$

and $o(\alpha)$means that $\frac{o(\alpha)}{\alpha}\to 0$ at $\alpha\to 0$.

Taking into account expansion (8) in formula (7), we obtain that

$$\Delta S(u)=\sum_{i=1}^{k}\frac{\partial\varphi'\big(z(T_1,X_1),z(T_2,X_2),\dots,z(T_k,X_k)\big)}{\partial a_i}\Delta z(T_i,X_i)+$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1}\psi'(t,x)\Delta z(t,x)dxdt-$$

$$-\int_{t_0}^{t}\int_{x_0}^{x}\left[\int_{t_0}^{t_1}\int_{x_0}^{x_1}\psi'(\tau,s)\big[f\big(\tau,s,t,x,\bar{z}(t,x),\bar{u}(t,x)\big)-f\big(\tau,s,t,x,z(t,x),u(t,x)\big)\big]dxdt\right]dsd\tau+$$

$$+o_1\left(\sum_{i=1}^{k}\|\Delta z(T_i,X_i)\|\right). \quad (9)$$

From formula (4), denoting characteristic function of the rectangle $[t_0,T_i]\times[x_0,X_i]$ by $\alpha_i(t,x)$, we obtain that

$$\Delta z(T_i, X_i) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \alpha_i(t,x)\left[f\left(T_i, X_i, t, x, \bar{z}(t,x), \bar{u}(t,x)\right) - \right.$$

$$\left. - f\left(T_i, X_i, t, x, z(t,x), u(t,x)\right)\right] dxdt. \qquad (10)$$

Taking into account expression (10) for $\Delta z(T_i, X_i)$ in (9), the formula for representing the increment of the quality functional is represented in the form

$$\Delta S(u) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \alpha_i(t,x) \sum_{i=1}^{k} \frac{\partial \varphi'\left(z(T_1,X_1), z(T_2,X_2), \dots, z(T_k,X_k)\right)}{\partial a_i} \times$$

$$\times \left[f\left(T_i, X_i, t, x, \bar{z}(t,x), \bar{u}(t,x)\right) - f\left(T_i, X_i, t, x, z(t,x), u(t,x)\right)\right] dxdt +$$

$$+ \int_{t_0}^{t_1} \int_{x_0}^{x_1} \psi'(t,x) \Delta z(t,x) dxdt -$$

$$- \int_{t_0}^{t} \int_{x_0}^{x} \left[\int_{t}^{t_1} \int_{x}^{x_1} \psi'(\tau,s)\left[f\left(\tau, s, t, x, \bar{z}(t,x), \bar{u}(t,x)\right) - f\left(\tau, s, t, x, z(t,x), u(t,x)\right)\right] dxdt \right] dsd\tau +$$

$$+ o_1\left(\sum_{i=1}^{k} \|\Delta z(T_i, X_i)\|\right). \qquad (11)$$

Let us introduce an analogue of the Pontryagin function for the problem under investigation in the form

$$H(t,x,z,u,\psi) = \int_{t}^{t_1} \int_{x}^{x_1} \psi'(\tau,s) f\left(\tau, s, t, x, z(t,x), u(t,x)\right) dsd\tau -$$

$$- \sum_{i=1}^{k} \alpha_i(t,x) \frac{\partial \varphi'\left(z(T_1,X_1), z(T_2,X_2), \dots, z(T_k,X_k)\right)}{\partial a_i} f(T_i, X_i, t, x, z, u).$$

In this case, the formula for increment (11) of the quality criterion takes the form

$$\Delta S(u) = - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left[H\left(t, x, \bar{z}(t,x), \bar{u}(t,x), \psi(t,x)\right) - H\left(t, x, z(t,x), u(t,x), \psi(t,x)\right)\right] dxdt +$$

$$+ \int_{t_0}^{t_1} \int_{x_0}^{x_1} \psi'(t,x) \Delta z(t,x) dxdt + o_1\left(\sum_{i=1}^{k} \|\Delta z(T_i, X_i)\|\right). \qquad (12)$$

Applying the Taylor formula to the difference

$$H(t,x,\bar{z},\bar{u},\psi) - H(t,x,z,u,\psi),$$

we arrive at the expansion

$$H\left(t, x, \bar{z}(t,x), \bar{u}(t,x), \psi(t,x)\right) - H\left(t, x, z(t,x), u(t,x), \psi(t,x)\right) =$$

$$= H'_z\big(t,x,z(t,x),\bar u(t,x),\psi(t,x)\big) + o_2(\|\Delta z(t,x)\|). \quad (13)$$

Further, taking into account expansion (13) in (12) and grouping similar terms after some transformations, we will have

$$\Delta S(u) = \int_{t_0}^{t_1}\int_{x_0}^{x_1} \psi'(t,x)\Delta z(t,x)dxdt - \int_{t_0}^{t_1}\int_{x_0}^{x_1} H'_z\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\Delta z(t,x)dxdt -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \big[H\big(t,x,z(t,x),\bar u(t,x),\psi(t,x)\big) - H\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\big]dxdt -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \big[H'_z\big(t,x,z(t,x),\bar u(t,x),\psi(t,x)\big) - H'_z\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\big]\Delta z(t,x)dxdt +$$

$$+o_1\left(\sum_{i=1}^{k}\|\Delta z(T_i,X_i)\|\right) - \int_{t_0}^{t_1}\int_{x_0}^{x_1} o_2(\|\Delta z(t,x)\|)dxdt. \quad (14)$$

Assuming that the vector function $\psi(t,x)$ satisfies the relation

$$\psi(t,x) = H_z\big(t,x,z(t,x),u(t,x),\psi(t,x)\big). \quad (15)$$

Then from (14) we obtain that

$$\Delta S(u) = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} \big[H\big(t,x,z(t,x),\bar u(t,x),\psi(t,x)\big) - H\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\big]dxdt -$$

$$-\int_{t_0}^{t_1}\int_{x_0}^{x_1} \big[H'_z\big(t,x,z(t,x),\bar u(t,x),\psi(t,x)\big) - H'_z\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\big]\Delta z(t,x)dxdt +$$

$$+o_1\left(\sum_{i=1}^{k}\|\Delta z(T_i,X_i)\|\right) - \int_{t_0}^{t_1}\int_{x_0}^{x_1} o_2(\|\Delta z(t,x)\|)dxdt. \quad (16)$$

Relation (15) is a linear inhomogeneous Volterra integral equation, which, following, e.g., [1-3], we call the conjugate system for the considered control problem.

The constructed formula for increment (16) makes it possible to establish the necessary optimality condition.

Proceeding from (4) to the norm and using the triangle rule, after some transformations we obtain that

$$\|\Delta z(t,x)\| \le K_1\left[\int_{t_0}^{t}\int_{x_0}^{x} \big\|f\big(t,x,\tau,s,z(\tau,s),\bar u(\tau,s)\big) - \big(t,x,\tau,s,z(\tau,s),u(\tau,s)\big)\big\|dsd\tau + \right.$$

$$\left. + \int_{t_0}^{t}\int_{x_0}^{x} \|\Delta z(\tau,s)\|dsd\tau\right],$$

where $K_1 = const > 0$ is some constant.

Applying the Gronwall-Wendorff lemma to the last inequality (see, e.g., [6]), we arrive at the required estimate:

$$\|\Delta z(t,x)\| \le K_2 \int_{t_0}^{t_1}\int_{x_0}^{x_1} \left\| f\big(t_1,x_1,t,x,z(t,x),\bar{u}(t,x)\big) - f\big(t_1,x_1,t,x,z(t,x),u(t,x)\big)\right\| dxdt,$$

$$K_2 = const > 0. \tag{17}$$

Suppose $(\theta,\xi) \in [t_0,t_1) \times [x_0,x_1)$ is an arbitrary regular point (Lebesgue point) (see, e.g., [1,6]) of the control $u(t,x), v \in U$ is an arbitrary vector, and $\varepsilon > 0$ is an arbitrary sufficiently small number such that $\theta + \varepsilon < t_1, \xi + \varepsilon < x_1$.

Special increment of the admissible control $u(t,x)$ will be determined from the formula

$$\Delta u_\varepsilon(t,x) = \begin{cases} v - u(t,x), (t,x) \in D_\varepsilon = [\theta,\theta+\varepsilon) \times [\,\xi,\xi+\varepsilon), \\ 0, \qquad\qquad\qquad (t,x) \in D \backslash D_\varepsilon. \end{cases} \tag{18}$$

Denote by $\Delta z_\varepsilon(t,x)$ the special increment of the state $z(t,x)$ corresponding to the special increment (18) of the control $u(t,x)$.

It follows from estimate (17) that

$$\|\Delta z_\varepsilon(t,x)\| \le K_3\varepsilon^2, (t,x) \in D, \tag{19}$$

where $K_3 = const > 0$ is some constant.

Taking into account (18), (19) from (16), by the mean value theorem, we obtain a special increment of the quality functional in the form

$$S(u+\Delta u_\varepsilon) - S(u) =$$

$$= -\varepsilon^2\big[H\big(\theta,\xi,z(\theta,\xi),v,\psi(\theta,\xi)\big) - H\big(\theta,\xi,z(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi)\big)\big] + o\,(\varepsilon^2). \tag{20}$$

By virtue of arbitrariness and sufficient smallness of $\varepsilon > 0$, expansion (20) implies the following statement.

Theorem 1. The optimality of the admissible control $u(t,x)$ requires that the relation

$$\max_{v \in U} H\big(\theta,\xi,z(\theta,\xi),v,\psi(\theta,\xi)\big) = H\big(\theta,\xi,z(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi)\big) \tag{21}$$

hold for all $(\theta,\xi) \in [t_0,t_1) \times [x_0,x_1)$.

Relation (21) is an analogue of Pontryagin's maximum condition in the problem under investigation, being a first-order necessary optimality condition.

Under some additional assumptions, we can obtain a linearized necessary optimality condition.

4. Linearized maximum principle

Suppose the set U is convex, and $f(t,x,\tau,s,z,u)$ is continuousin the totality of variables together with partial derivatives with respect to (z,u). Then we can write that

$$H(t,x,\bar{z},\bar{u},\psi) - H(t,x,z,u,\psi) =$$

$$= H'_z(t,x,z,u,\psi)\Delta z + H'_u(t,x,z,u,\psi)\Delta u + o_3(\|\Delta z\| + \|\Delta u\|). \tag{22}$$

Taking into account the expansion (22), in increment formula (12) and taking into account (15), we obtain that

$$\Delta S(u) = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} H'_u\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\Delta u(t,x)dxdt +$$

$$+o_1\left(\sum_{i=1}^{k}\|\Delta z(T_i,X_i)\|\right) + \int_{t_0}^{t_1}\int_{x_0}^{x_1} o_3(\|\Delta z(t,x)\| + \|\Delta u(t,x)\|)dxdt. \quad (23)$$

From (4) it turns out that

$$\|\Delta z(t,x)\| \le K_4\left[\int_{t_0}^{t}\int_{x_0}^{x}\|\Delta u(\tau,s)\|dsd\tau + \int_{t_0}^{t}\int_{x_0}^{x}\|\Delta z(\tau,s)\|dsd\tau\right], \quad (24)$$

where $K_4 = const > 0$ is some constant.

Applying the Gronwall-Wendorff lemma to inequality (24), we obtain the estimate

$$\|\Delta z(t,x)\| \le K_5\int_{t_0}^{t_1}\int_{x_0}^{x_1}\|\Delta u(\tau,s)\|dsd\tau, \quad (25)$$

where $K_5 = const > 0$ is some constant.

Assuming that $\mu \in [0,1]$ is an arbitrary number, and $v(t,x) \in U, (t,x) \in D$ is an arbitrary admissible control, the special increment of the control $u(t,x)$ will be determined from the formula

$$\Delta u_\mu(t,x) = \mu[v(t,x) - u(t,x)], (t,x) \in D. \quad (26)$$

This is possible due to the convexity of the set U.

Suppse $\Delta z_\mu(t,x)$ is a special increment of the state $z(t,x)$, corresponding to special increment (26) of the control. It follows from estimate (25) that

$$\|\Delta z_\mu(t,x)\| \le K_5\mu\int_{t_0}^{t_1}\int_{x_0}^{x_1}\|v(\tau,s) - u(\tau,s)\|dsd\tau. \quad (27)$$

Taking into account (26) and (27) in (23), we obtain that

$$S(u+\Delta u_\mu) - S(u) =$$

$$= -\mu\int_{t_0}^{t_1}\int_{x_0}^{x_1} H'_u\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\big(v(t,x) - u(t,x)\big)dxdt. \quad (28)$$

From expansion (28) follows

Theorem 2. If the set U is convex, then the optimality of the admissible control $u(t,x)$ requires that the inequality

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} H'_u\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\big(v(t,x) - u(t,x)\big)dxdt \le 0, \quad (29)$$

hold for all $v(t,x) \in U, (t,x) \in D$.

The proved necessary optimality condition (29) is an analogue of the linearized integral maximum condition. Using the scheme, e.g., from [15], a pointwise necessary optimality condition is proved.

Corollary. If the set U is convex, then the optimality of the admissible control $u(t,x)$ requires that the condition

$$\max_{v\in U} H_u'\big(\theta,\xi,z(\theta,\xi),v,\psi(\theta,\xi)\big) = H_u'\big(\theta,\xi,z(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi)\big) \tag{30}$$

hold for all $v(\theta,\xi) \in [t_0,t_1) \times [x_0,x_1)$.

Inequality (30) is an analogue of the linearized (integral) [2] maximum principle.

4. An analogue of the Euler equation

Suppose the set U in the problem under investigation is open, and ε is an an arbitrary number sufficiently small in absolute value. Under the assumptions made, the special increment of the admissible control can be determined from the formula

$$\Delta u_\varepsilon(t,x) = \varepsilon\delta u(t,x), (t,x) \in D, \tag{31}$$

where $\delta u(t,x) \in R^r, (t,x) \in D$ is an arbitrary measurable and bounded r-dimensional vector-function (admissible variation of the control $u(t,x)$).

Taking into account estimate (24), as well as formula (31) in (23), we arrive at the expansion

$$S(u+\varepsilon\delta u) - S(u) =$$
$$= -\varepsilon \int_{t_0}^{t_1}\int_{x_0}^{x_1} H_u'\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\delta u(t,x)dxdt + o(\varepsilon). \tag{32}$$

It follows from expansion (32) that the first variation (in the classical sense) of functional (1) has the form:

$$\delta^1 S(u;\delta u) = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} H_u'\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\delta u(t,x)dxdt. \tag{33}$$

From (33), based on the main result of the classical calculus of variations (see, e.g., [16, 17]) it follows that if $u(t,x)$ is an optimal control, then for all admissible variations $\delta u(t,x)$ of the control $u(t,x)$, the following identity takes place:

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} H_u'\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\delta u(t,x)dxdt = 0. \tag{34}$$

Identity (34) is an implicit necessary first-order optimality condition. From it, we can obtain the necessary optimality condition in the form of the Euler equation [16,17].

We have

Theorem 3. If the set U is open, then the optimality of the admissible control $u(t,x)$ requires that the relation

$$H_u\big(\theta,\xi,z(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi)\big) = 0 \tag{35}$$

hold for all $(\theta,\xi) \in [t_0,t_1) \times [x_0,x_1)$.

Proof. Let us assume the opposite. Suppose there is a point $(\bar{\theta},\bar{\xi}) \in [t_0,t_1] \times [x_0,x_1]$ and a numbers s $(1 \le s \le n)$ such as

$$H_{u_s}\left(\bar{\theta},\bar{\xi},z(\bar{\theta},\bar{\xi}),u(\bar{\theta},\bar{\xi}),\psi(\bar{\theta},\bar{\xi})\right)=\alpha\neq 0.$$

Now the coordinates of the admissible variation $\delta u(t,x)=(\delta u_1,\delta u_2,\dots,\delta u_r)'$ will be determined as follows:

$$\delta u_i(t,x)=0,(t,x)\in D, i\neq s,$$

$$\delta u_s(t,x)=H_{u_s}\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\,, \tag{36}$$

at $(t,x)\in[\bar{\theta},\bar{\theta}+\varepsilon)\times\left[\,\bar{\xi},\bar{\xi}+\varepsilon\right)$.

$$\delta u_s(t,x)=0,(t,x)\in D\backslash[\bar{\theta},\bar{\theta}+\varepsilon)\times\left[\,\bar{\xi},\bar{\xi}+\varepsilon\right)$$

where $\varepsilon>0$ is an arbitrary sufficiently small number.

Then we get that if the variation $\delta u(t,x)$ of the control $u(t,x)$ is determined from formula (36), then we get that

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} H_u'\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\delta u(t,x)dxdt=$$

$$=\int_{\theta}^{\theta+\varepsilon}\int_{\xi}^{\xi+\varepsilon}\left[\frac{\partial H\big(t,x,z(t,x),u(t,x),\psi(t,x)\big)\delta u(t,x)}{\partial u_s}\right]^2=\varepsilon^2\alpha^2+o(\varepsilon^2)\neq 0.$$

This contradicts the optimality condition (34). We get a contradiction. This proves the theorem.

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5. Conclusion

In the problem under investigation, the multipoint nature of the quality functional complicates its study. Applying one version of the increment method, by means of the constructed formulas for increments of the quality criterion under various assumptions, a number of necessary optimality conditions of a constructive nature have been established.

References

[1] Л.С. Понтрягин, В.Г. Болтянский, Р.В. Гамкрелидзе, Е.Ф. Мищенко, Математическая теория оптимальных процессов, Москва, Наука, (1984) 386 с. [In Russian: L.S. Pontryagin, V.G. Boltyansky, R.V. Gamkrelidze, E.F. Mishchenko, Mathematical theory of optimal processes, Moscow, Nauka]

[2] Р. Габасов, Ф.М. Кириллова, Принцип максимума в теории оптимального управления, Москва, «Либроком», (2011) 272 с. [In Russian: R. Gabasov, F.M. Kirillova, The maximum principle in optimal control theory, Moscow, Librokom]

[3] Ф.П. Васильев, Методы оптимизации, Москва, Факториал, (2002) 824 с. [In Russian: F.P. Vasilyev, Optimization methods, Moscow, Faktorial]

[4] К.Б. Мансимов, Особые управления в системами с запаздыванием, Баку, ЭЛМ, (1999) 176 с. [In Russian: K.B. Mansimov, Singular controls in systems with delay, Baku, ELM]

[5] А.И. Егоров, Л.Н. Знаменская, Введение в теорию управления системами с распределенными параметрами, СПБ, (2017) 292 с. [In Russian: A.I. Yegorov, L.N. Znamenskaya, Introduction to the theory of control of systems with distributed parameters, SPB]

[6] М.М. Новоженов, В.И. Сумин, М.М. Сумин, Методы оптимального управления системами математической физики, Горький, ГГУ, (1986) 87 с. [In Russian: M.M. Novozhenov, V.I. Sumin, M.M. Sumin, Methods of optimal control of systems of mathematical physics, Gorky, GSU]

[7] К.Б. Мансимов, Условия оптимальности второго порядка в системах Гурса-Дарбу при наличии ограничений, Дифферен. уравнения. 6 (1990) с.954-965. [In Russian: K.B. Mansimov, Second-order optimality conditions in Goursat-Darboux systems with constraints, Differen. uravneniya]
[8] В.А. Срочко, Вариационный принцип максимума и методы линеаризации в задачах оптимального управления, Иркутск, Изд-во ИГУ, (1989) 160 с. [In Russian: V.A. Srochko, The variational maximum principle and linearization methods in optimal control problems, Irkutsk, Irkutsk State University]
[9] К.Б. Мансимов, К оптимальности особых, в классическом смысле, управлений в системах Гурса-Дарбу, Докл. АН СССР. 286 No.4 (1986) с.808-812. [In Russian: K.B. Mansimov, On the optimality of special, in the classical sense, controls in Goursat-Darboux systems, Dokl. AN SSSR]
[10] Т.К. Ахмедов, Т.О. Нуримов, О двумерных интегральных уравнениях и неравенствах типа Вольтерра, Докл. АН Уз. ССР. No.3 (1972) с.3-5. [In Russian: T.K. Ahmedov, T.O. Nurimov, On two-dimensional integral equations and inequalities of Volterra type, Dokl. AN Uz. SSR]
[11] А.Б. Васильева, А.Б. Тихонов, Интегральные уравнения, Москва, Изд-во МГУ, (1989) 156 с. [In Russian: A.B. Vasilyeva, A.B. Tikhonov, Integral equations, Moscow, Moscow State University]
[12] В. Вольтерра, Теория функционалов, интегральных и интегро-дифференциальных уравнений, Москва, Наука, (1992) 304 с. [In Russian: V. Volterra, Theory of functionals, integral and integro-differential equations, Moscow, Nauka]
[13] И.Г. Петровский, Лекции по теории интегральных уравнений, Москва, Изд-во МГУ, (1989) 136 с. [In Russian: I.G. Petrovsky, Lectures on the theory of integral equations, Moscow, Moscow State University]
[14] А.А. Абдуллаев, К.Б. Мансимов, Необходимые условия оптимальности в процессах, описываемых системой интегральных уравнений типа Вольтерра, Баку, ЭЛМ, (2013) 224 с. [In Russian: A.A. Abdullayev, K.B. Mansimov, Necessary optimality conditions in processes described by a system of Volterra integral equations, Baku, ELM]
[15] K.B. Mənsimov, qursa-Darbu sistemləri ilə optimal idarəetmə məsələləri, Bakı, BDU nəşriyyatı, (1998) 110 s. [In Azerbaijani: K.B. Mansimov, Optimal control issues with Goursat-Darboux systems, Baku, BSU]
[16] Р. Габасов, Ф.М. Кирилова, Особые оптимальное управления, Москва, «Либроком», (2011) 256 с. [In Russian: R. Gabasov, F.M. Kirillova, Singular optimal controls, Moscow, Librokom]
[17] Р. Габасов, Ф.М. Кирилова, В.В. Альсевич, и др., Методы оптимизации, Минск, Изд.-во «Четыре четверта», (2011) 472 с. [In Russian: R. Gabasov, F.M. Kirilova, V.V. Alsevich, et al, Optimization methods, Minsk, Chetyre Chetverta]