

A necessary optimality condition of the Pontryagin maximum principle type in one problem of optimal control of a system with distributed parameters

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ARTICLE INFO	ABSTRACT
<hr/> <i>Article history:</i> Received 28.09.2022 Received in revised form 16.10.200 Accepted 23.10.2022 Available online 05.04.2023 <hr/> <i>Keywords:</i> Hyperbolic integro-differential equation Volterra integral equation Pontryagin maximum principle Necessary optimality condition Conjugate equation system <hr/>	<hr/> <i>We consider a variable-structure optimal control problem described in different domains by a hyperbolic integro-differential equation and a Volterra integral equation, respectively. The quality functional is terminal. A formula for the increment of the quality criterion is constructed and an analogue of L.S. Pontryagin's maximum principle is proved by investigating on special McShane-type variations.</i> <hr/>

1. Introduction

In [1-4] etc., a number of multistage optimal control problems described in different time intervals by different equations are investigated.

Such optimal control problems are also called variable-structure optimal control problems (see e.g. [1-6]).

In this study, one problem of optimal control of a variable-structure object described in different domains by hyperbolic integro-differential equations and Volterra integral equations is considered.

An analogue of L.S. Pontryagin's maximum principle is established.

2. Problem statement

Suppose that $D_i = [t_{i-1}, t_i] \times [x_0, x_1], i = 1, 2$ are specified rectangles, with $t_0 < t_1 < t_2; x_0 < x_1 < x_2$.

Suppose that the controlled process in the domain $D = D_1 \cup D_2$ is described by the boundary problem

$$\frac{\partial^2 z_1(t, x)}{\partial t \partial x} = \int_{t_0}^t \int_{x_0}^x f_1(t, x, \tau, s, z_1(\tau, s), u_1(\tau, s)) ds d\tau, (t, x) \in D_1, \quad (1)$$

$$\begin{aligned} z_1(t_0, x) &= a(x), x \in [x_0, x_1], \\ z_1(t, x_0) &= b(t), t \in [t_0, t_1] \end{aligned} \quad (2)$$

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and the system of two-dimensional Volterra integral equations

$$z_2(t, x) = \int_{t_1}^t \int_{x_0}^x f_2(t, x, \tau, s, z_2(\tau, s), u_2(\tau, s)) ds d\tau + G(z_1(t_1, x)), (t, x) \in D_2. \quad (3)$$

Here, $f_i(t, x, \tau, s, z_i, u_i), i = 1, 2$ are specified n -dimensional vector functions continuous in the set of variables with partial derivatives in $z_i, i = 1, 2, a(x)$ and $b(t)$ are specified n -dimensional absolutely continuous vector functions, $G(z_1)$ is specified n -dimensional continuously differentiable vector function, $U_1 \subset R^r, U_2 \subset R^q$ are specified non-empty and bounded sets, $u_1(t, x)(u_2(t, x))$ is a $r(q)$ -dimensional is a measurable and bounded vector function of control actions, satisfying the constraint

$$u_1(t, x) \in U_1 \subset R^r, (t, x) \in D_1. \quad (4)$$

$$u_2(t, x) \in U_2 \subset R^q, (t, x) \in D_2. \quad (5)$$

We shall call the pair $(u_1(t, x), u_2(t, x))$ with the above properties an admissible control.

Suppose that a unique, absolutely continuous solution $z_1(t, x)$ of boundary value problem (1)-(2) and a unique continuous solution $z_2(t, x)$ of integral equation (3) correspond to each given admissible control $(u_1(t, x), u_2(t, x))$.

On the solutions of problem (1)-(3) given by all possible admissible controls $(u_1(t, x), u_2(t, x))$, we shall determine the terminal functional

$$J(u_1, u_2) = \varphi_1(z_1(t_1, x)) + \varphi_2(z_2(t_2, x)). \quad (6)$$

where $\varphi_i(z_i), i = 1, 2$ are specified continuously differentiable scalar functions.

The problem is to find the minimum value of terminal functional (6) under constraints (1)-(5).

The admissible control $(u_1(t, x), u_2(t, x))$ that satisfies a minimum value to functional (6) shall be called the optimal control, and the corresponding process $(u_1(t, x), u_2(t, x), z_1(t, x), z_2(t, x))$ shall be called the optimal process.

The aim of the study is to derive a necessary optimality condition of the Pontryagin maximum principle type in the problem under consideration.

3. The formula for the increment of the quality functional

Suppose that $(u_1(t, x), u_2(t, x), z_1(t, x), z_2(t, x))$ and $(\bar{u}_1(t, x) = u_1(t, x) + \Delta u_1(t, x), \bar{u}_2(t, x) = u_2(t, x) + \Delta u_2(t, x), \bar{z}_1(t, x) = z_1(t, x) + \Delta z_1(t, x), \bar{z}_2(t, x) = z_2(t, x) + \Delta z_2(t, x))$ are some admissible processes. It is clear then that $(\Delta z_1(t, x), \Delta z_2(t, x))$ will be the solution of the problem

$$\frac{\partial^2 \Delta z_1(t, x)}{\partial t \partial x} = \int_{t_0}^t \int_{x_0}^x [f_1(t, x, \tau, s, \bar{z}_1(\tau, s), \bar{u}_1(\tau, s)) - f_1(t, x, \tau, s, z_1(\tau, s), u_1(\tau, s))] ds d\tau, \quad (7)$$

$$\Delta z_1(t_0, x) = 0,$$

$$\Delta z_1(t, x_0) = 0. \quad (8)$$

$$\Delta z_2(t, x) = \int_{t_1}^t \int_{x_0}^x [f_2(t, x, \tau, s, \bar{z}_2(\tau, s), \bar{u}_2(\tau, s)) - f_2(t, x, \tau, s, z_2(\tau, s), u_2(\tau, s))] ds d\tau + G(\bar{z}_1(t_1, x)) - G(z_1(t_1, x)), \quad (9)$$

and the increment of functional (6) has the form

$$J(\bar{u}_1, \bar{u}_2) - J(u_1, u_2) = [\varphi_1(\bar{z}_1(t_1, x)) - \varphi_1(z_1(t_1, x))] + [\varphi_2(\bar{z}_2(t_2, x)) - \varphi_2(z_2(t_2, x))]. \quad (10)$$

Suppose that $\psi_i(t, x), i = 1, 2$ are as yet unknown vector functions.

From identities (7) and (9), we have that

$$\int_{t_0}^{t_1} \int_{x_0}^{x_1} \psi_1'(t, x) \frac{\partial^2 \Delta z_1(t, x)}{\partial t \partial x} dx dt = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \psi_1'(t, x) \left[\int_{t_0}^t \int_{x_0}^x [f_1(t, x, \tau, s, \bar{z}(\tau, s), \bar{u}(\tau, s)) - f_1(t, x, \tau, s, z(\tau, s), u(\tau, s))] ds d\tau \right] dx dt, \tag{11}$$

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{x_0}^{x_1} \psi_2'(t, x) \Delta z_2(t, x) dx dt = \\ & = \int_{t_1}^{t_2} \int_{x_0}^{x_1} \psi_2'(t, x) \left[\int_{t_1}^t \int_{x_0}^x [f_2(t, x, \tau, s, \bar{z}_2(\tau, s), \bar{u}_2(\tau, s)) - f_2(t, x, \tau, s, z_2(\tau, s), u_2(\tau, s))] ds d\tau \right] dx dt + \\ & + \int_{t_1}^{t_2} \int_{x_0}^{x_1} \psi_2'(t, x) [G(\bar{z}_1(t_1, x)) - G(z_1(t_1, x))] dx dt, \end{aligned} \tag{12}$$

From formula (11), given boundary conditions (8) and applying the Fubini formula (see e.g., [7]), we have that

$$\begin{aligned} & \psi_1'(t_1, x_1) \Delta z_1(t_1, x_1) - \int_{t_0}^{t_1} \frac{\partial \psi_1'(t, x)}{\partial t} \Delta z_1(t_1, x) dt - \int_{x_0}^{x_1} \frac{\partial \psi_1'(t, x)}{\partial x} \Delta z_1(t, x_1) dx + \\ & + \int_{t_0}^{t_1} \int_{x_0}^{x_1} \frac{\partial^2 \psi_1'(t, x)}{\partial t \partial x} \Delta z_1(t, x) dx dt = \\ & = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left[\int_t^{t_1} \int_x^{x_1} \psi_1'(\tau, s) [f_1(\tau, s, t, x, \bar{z}_1(t, x), \bar{u}_1(t, x)) - f_1(\tau, s, t, x, z_1(t, x), u_1(t, x))] ds d\tau \right] dx dt. \end{aligned} \tag{13}$$

And from identity (12) by introducing the notation

$$M(\psi_2(t, x), z_1) = \psi_2'(t, x) G(z_1)$$

and applying the Fubini formula, we have that

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{x_0}^{x_1} \psi_2'(t, x) \Delta z_2(t, x) dx dt = \\ & = \int_{t_1}^{t_2} \int_{x_0}^{x_1} \left[\int_t^{t_2} \int_x^{x_1} \psi_2'(t, x) [f_2(\tau, s, t, x, \bar{z}_2(t, x), \bar{u}_2(t, x)) - f_2(\tau, s, t, x, z_2(t, x), u_2(t, x))] ds d\tau \right] dx dt + \\ & + \int_{t_1}^{t_2} \int_{x_0}^{x_1} [M(\psi_2(t, x), \bar{z}_1(t_1, x)) - M(\psi_2(t, x), z_1(t_1, x))] dx dt, \end{aligned} \tag{14}$$

From the formula for increment (10) of the quality functional, using the Taylor formula, we obtain

$$\begin{aligned} J(\bar{u}_1, \bar{u}_2) - J(u_1, u_2) & = \frac{\partial \varphi_1'(z_1(t_1, x_1))}{\partial z_1} \Delta z_1(t_1, x_1) + \frac{\partial \varphi_2'(z_2(t_1, x_1))}{\partial z_2} \Delta z_2(t_2, x_1) + \\ & + o_1(\|\Delta z_1(t_1, x_1)\|) + o_2(\|\Delta z_2(t_1, x_1)\|). \end{aligned} \tag{15}$$

Here and further $\|\alpha\|$ means the norm of the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)'$ determined by the formula $\|\alpha\| = \sum_{i=1}^n |\alpha_i|$, $o(\alpha)$ is a quantity of a higher order than α , i.e., $\frac{o(\alpha)}{\alpha} \rightarrow 0$ at $\alpha \rightarrow 0$, and the dash (') is a scalar product operation for vectors, and a transpose operation for matrices.

Given identity (9) we can write that

$$\frac{\partial \varphi'_2(z_2(t_1, x_1))}{\partial z_2} \Delta z_2(t_2, x_1) = \frac{\partial \varphi'_2(z_2(t_1, x_1))}{\partial z_2} \int_{t_1}^{t_2} \int_{x_0}^{x_1} [f_2(t_2, x_1, t, x, \bar{z}_2(t, x), \bar{u}_2(t, x)) - f_2(t_2, x_1, t, x, z_2(t, x), u_2(t, x))] dxdt + \frac{\partial \varphi'_2(z_2(t_1, x_1))}{\partial z_2} [G(\bar{z}_1(t_1, x_1)) - G(z_1(t_1, x_1))].$$

We introduce the notation

$$N(z_1) = \frac{\partial \varphi'_2(z_2(t_1, x_1))}{\partial z_2} G(z_1),$$

$$H_1(x, z_1(t, x), u_1(t, x), \psi_1(t, x)) = \int_t^{t_1} \int_x^{x_1} \psi_1'(\tau, s) f_1(\tau, s, t, x, z_1(t, x), u_1(t, x)) dsd\tau,$$

$$H_2(x, z_2(t, x), u_2(t, x), \psi_2(t, x)) = -\frac{\partial \varphi'_2(z_2(t_1, x_1))}{\partial z_2} f_2(t_2, x_1, t, x, z_2(t, x), u_2(t, x)) + \int_t^{t_2} \int_x^{x_1} \psi_2'(\tau, s) f_2(\tau, s, t, x, z_2(t, x), u_2(t, x)) dsd\tau.$$

Using the Taylor formula, we obtain that

$$N(\bar{z}_1(t_1, x_1)) - N(z_1(t_1, x_1)) = \frac{\partial N'(z_1(t_1, x_1))}{\partial z_1} \Delta z_1(t_1, x_1) + o_3(\|z_1(t_1, x_1)\|), \quad (16)$$

$$M(\psi_2(t, x), \bar{z}_1(t_1, x)) - M(\psi_2(t, x), z_1(t_1, x)) = \frac{\partial M'(\psi_2(t, x), z_1(t_1, x))}{\partial z_1} \Delta z_1(t_1, x_1) + o_4(t; \|z_1(t_1, x)\|). \quad (17)$$

Taking into account the introduced notations and formulas (13), (14), (16), (17), increment (15) of the functional is represented in the form

$$\begin{aligned} J(\bar{u}_1, \bar{u}_2) - J(u_1, u_2) &= \frac{\partial \varphi'_1(z_1(t_1, x_1))}{\partial z_1} \Delta z_1(t_1, x_1) + \psi_1'(t_1, x_1) \Delta z_1(t_1, x_1) - \\ &\int_{t_0}^{t_1} \frac{\partial \psi_1'(t, x)}{\partial t} \Delta z_1(t_1, x) dt - \int_{x_0}^{x_1} \frac{\partial \psi_1'(t, x_1)}{\partial x} \Delta z_1(t, x_1) dx + \int_{t_0}^{t_1} \int_{x_0}^{x_1} \frac{\partial^2 \psi_1'(t, x)}{\partial t \partial x} \Delta z_1(t, x) dxdt + \\ &+ \frac{\partial N'(z_1(t_1, x_1))}{\partial z_1} \Delta z_1(t_1, x_1) - \int_{t_1}^{t_2} \int_{x_0}^{x_1} \frac{\partial M'(\psi_2(t, x), z_1(t_1, x))}{\partial z_1} \Delta z_1(t, x) dxdt + \\ &+ \int_{t_1}^{t_2} \int_{x_0}^{x_1} \psi_2'(t, x) \Delta z_2(t, x) dxdt - \\ &- \int_{t_0}^t \int_{x_0}^x [H_1(t, x, \bar{z}(t, x), \bar{u}(t, x), \psi_1(t, x)) - H_1(t, x, z(t, x), u(t, x), \psi_1(t, x))] dxdt - \\ &- \int_{t_1}^{t_2} \int_{x_0}^{x_1} [H_2(t, x, \bar{z}_2(t, x), \bar{u}_2(t, x), \psi_2(t, x)) - H_2(t, x, z_2(t, x), u_2(t, x), \psi_2(t, x))] dxdt + \\ &+ o_1(\|\Delta z_1(t_1, x_1)\|) + o_2(\|\Delta z_2(t_1, x_1)\|) + o_3(\|z_1(t_1, x_1)\|) - \\ &- \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_4(t; \|z_1(t_1, x)\|) dxdt. \end{aligned} \quad (18)$$

Taking into account the smoothness conditions imposed on the Hamilton-Pontryagin functions $H_i(t, x, z_i(t, x), u_i(t, x), \psi_i(t, x))$, $i = 1, 2$ in z_i , $i = 1, 2$, after some manipulations we obtain that

$$\begin{aligned} &H_i(t, x, \bar{z}_i(t, x), \bar{u}_i(t, x), \psi_i(t, x)) - H_i(t, x, z_i(t, x), u_i(t, x), \psi_i(t, x)) = \\ &= H_i(t, x, z_i(t, x), \bar{u}_i(t, x), \psi_i(t, x)) - H_i(t, x, z_i(t, x), u_i(t, x), \psi_i(t, x)) + \end{aligned}$$

$$+ \frac{\partial H_i'(t, x, z_i(t, x), u_i(t, x), \psi_i(t, x))}{\partial z_i} \Delta z_i(t, x) + \left[\frac{\partial H_i(t, x, z_i(t, x), \bar{u}_i(t, x), \psi_i(t, x))}{\partial z_i} - \frac{\partial H_i(t, x, z_i(t, x), u_i(t, x), \psi_i(t, x))}{\partial z_i} \right]' \Delta z_i(t, x) + o_{i+4}(\|\Delta z_i(t, x)\|), i = 1, 2.$$

Given this formula in increment formula (18), we will have

$$\begin{aligned} J(\bar{u}_1, \bar{u}_2) - J(u_1, u_2) &= \frac{\partial \varphi_1'(z_1(t_1, x_1))}{\partial z_1} \Delta z_1(t_1, x_1) + \psi_1'(t_1, x_1) \Delta z_1(t_1, x_1) - \\ &\quad - \int_{t_0}^{t_1} \frac{\partial \psi_1'(t, x)}{\partial t} \Delta z_1(t, x) dt - \int_{x_0}^{x_1} \frac{\partial \psi_1'(t, x)}{\partial x} \Delta z_1(t, x) dx + \\ &\quad + \int_{t_0}^{t_1} \int_{x_0}^{x_1} \frac{\partial^2 \psi_1'(t, x)}{\partial t \partial x} \Delta z_1(t, x) dx dt + \frac{\partial N'(z_1(t_1, x_1))}{\partial z_1} \Delta z_1(t_1, x_1) - \\ &\quad - \int_{t_1}^{t_2} \int_{x_0}^{x_1} \frac{\partial M'(\psi_2(t, x), z_1(t_1, x))}{\partial z_1} \Delta z_1(t, x) dx dt + \int_{t_1}^{t_2} \int_{x_0}^{x_1} \psi_2'(t, x) \Delta z_2(t, x) dx dt - \\ &\quad - \int_{t_0}^{t_1} \int_{x_0}^{x_1} [H_1(t, x, \bar{z}_1(t, x), \bar{u}_1(t, x), \psi_1(t, x)) - H_1(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))] dx dt - \\ &\quad - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \frac{\partial H_1'(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))}{\partial z_1} \Delta z_1(t, x) dx dt - \\ &\quad - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left[\frac{\partial H_1(t, x, z_i(t, x), \bar{u}_1(t, x), \psi_1(t, x))}{\partial z_1} - \frac{\partial H_1(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))}{\partial z_1} \right]' \Delta z_1(t, x) dx dt - \\ &\quad - \int_{t_1}^{t_2} \int_{x_0}^{x_1} [H_2(t, x, \bar{z}_2(t, x), \bar{u}_2(t, x), \psi_2(t, x)) - H_2(t, x, z_2(t, x), u_2(t, x), \psi_2(t, x))] dx dt - \\ &\quad - \int_{t_1}^{t_2} \int_{x_0}^{x_1} \frac{\partial H_2'(t, x, z_2(t, x), u_2(t, x), \psi_2(t, x))}{\partial z_2} \Delta z_2(t, x) dx dt - \\ &\quad - \int_{t_1}^{t_2} \int_{x_0}^{x_1} \left[\frac{\partial H_2(t, x, z_2(t, x), \bar{u}_2(t, x), \psi_2(t, x))}{\partial z_2} - \frac{\partial H_2(t, x, z_2(t, x), u_2(t, x), \psi_2(t, x))}{\partial z_2} \right]' \Delta z_2(t, x) dx dt - \\ &\quad + o_1(\|\Delta z_1(t_1, x_1)\|) + o_2(\|\Delta z_2(t_1, x_1)\|) + o_3(\|z_1(t_1, x_1)\|) - \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_4(t; \|z_1(t, x)\|) dx dt - \\ &\quad - \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_5(\|z_1(t, x)\|) dx dt - \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_6(\|z_1(t, x)\|) dx dt. \end{aligned} \tag{19}$$

Suppose that the vector function $\psi_i(t, x), i = \overline{0, p}$ satisfy the relations

$$\begin{aligned} \frac{\partial^2 \psi_i(t, x)}{\partial t \partial x} &= \frac{\partial H_1(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))}{\partial z_1}, \\ \frac{\partial \psi_1(t, x)}{\partial t} &= 0, \end{aligned} \tag{20}$$

$$\frac{\partial \psi_1(t_1, x)}{\partial x} = - \frac{\partial M(\psi_2(t, x), z_1(t_1, x))}{\partial z_1}, \tag{21}$$

$$\psi_1(t_1, x_1) = - \frac{\partial \varphi_1(z_1(t_1, x_1))}{\partial z_1} - \frac{\partial N(z_1(t_1, x_1))}{\partial z_1}, \tag{22}$$

$$\psi_2(t, x) = \frac{\partial H_2(t, x, z_2(t, x), u_2(t, x), \psi_2(t, x))}{\partial z_2}. \tag{23}$$

Relations (20)-(21) are a linear boundary value problem for $\psi_1(t, x)$, and relation (23) is a linear integral equation with respect to $\psi_2(t, x)$.

Following the classical terminology, we shall call them conjugate systems.

If relations (20)-(23) are fulfilled, increment formula (19) will take the form

$$\begin{aligned} J(\bar{u}_1, \bar{u}_2) - J(u_1, u_2) = & - \sum_{i=1}^2 \int_{t_{i-1}}^{t_i} \int_{x_0}^{x_1} [H_i(t, x, z_i(t, x), \bar{u}_i(t, x), \psi_i(t, x)) - \\ & - H_i(t, x, z_i(t, x), u_i(t, x), \psi_i(t, x))] dxdt - \\ & - \sum_{i=1}^2 \int_{t_{i-1}}^{t_i} \int_{x_0}^{x_1} \left[\frac{\partial H_i(t, x, z_i(t, x), \bar{u}_i(t, x), \psi_i(t, x))}{\partial z_i} - \right. \\ & \left. - \frac{\partial H_1(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))}{\partial z_1} \right] \Delta z_i(t, x) dxdt - \\ & + \sum_{i=1}^2 o_i(\|\Delta z_i(t_i, x_1)\|) + o_3(\|z_1(t_1, x_1)\|) - \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_4(t; \|z_1(t, x)\|) dxdt - \\ & - \sum_{i=1}^2 \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_{i+4}(\|z_i(t, x)\|) dxdt. \end{aligned} \tag{24}$$

Suppose that $\Delta u_2(t, x) = 0$. Then from relations (7)-(10) we obtain that

$$\Delta z_1(t, x) = \int_{t_0}^t \int_{x_0}^x [f_1(t, x, \tau, s, \bar{z}_1(\tau, s), \bar{u}_1(\tau, s)) - f_1(t, x, \tau, s, z_1(\tau, s), u_1(\tau, s))] dsd\tau, \tag{25}$$

$$\begin{aligned} \Delta z_2(t, x) = & \int_{t_2}^t \int_{x_0}^x [f_2(t, x, \tau, s, \bar{z}_2(\tau, s), \bar{u}_2(\tau, s)) - f_2(t, x, \tau, s, z_2(\tau, s), u_2(\tau, s))] dsd\tau + \\ & + G(\bar{z}_1(t_1, x_1)) - G(z_1(t_1, x_1)). \end{aligned} \tag{26}$$

Assuming that $\Delta u_1(t, x) = 0$, then from (7)-(9) and (10) we obtain that

$$\Delta z_1(t, x) = 0,$$

$$\Delta z_2(t, x) = \int_{t_2}^t \int_{x_0}^x [f_2(t, x, \tau, s, \bar{z}_2(\tau, s), \bar{u}_2(\tau, s)) - f_2(t, x, \tau, s, z_2(\tau, s), u_2(\tau, s))] dsd\tau. \tag{27}$$

From formula (25), applying the Gronwall-Wendorff lemma (see, e.g., [8]), after some manipulations we obtain the estimate

$$\begin{aligned} \|\Delta z_1(t, x)\| \leq & L_1 \int_{t_0}^t \int_{x_0}^x \left[\int_{\tau}^t \int_s^x \|f_1(t, x, \tau, s, z(\tau, s), \bar{u}(\tau, s)) - \right. \\ & \left. - f_1(t, x, \tau, s, z(\tau, s), u(\tau, s))\| dxdt \right] dsd\tau, \end{aligned} \tag{28}$$

where $L_1 = const > 0$ is a constant.

From relation (26), applying the Gronwall-Wendorff lemma, we arrive at the estimate

$$\|\Delta z_2(t, x)\| \leq L_2 \|\Delta z_1(t, x)\|, (t, x) \in D_2 \tag{29}$$

($L_2 = const > 0$ is a constant).

And from formula (27) we similarly obtain the estimate

$$\|\Delta z_2(t, x)\| \leq L_3 \int_{t_1}^t \int_{x_0}^x \|f_1(t, x, \tau, s, z(\tau, s), \bar{u}(\tau, s)) - f(t, x, \tau, s, z(\tau, s), u(\tau, s))\| ds d\tau. \quad (30)$$

($L_3 = const > 0$ is a constant).

4. Necessary optimality condition

Suppose that $\Delta u_2(t, x) = 0$. Then it follows from increment formula (24) that

$$\begin{aligned} J(\bar{u}_1, u_2) - J(u_1, u_2) = & - \int_{t_0}^{t_1} \int_{x_0}^{x_1} [H_1(t, x, z_1(t, x), \bar{u}_1(t, x), \psi_1(t, x)) - \\ & - H_1(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))] dx dt - \\ & - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \frac{\partial H_1'(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))}{\partial z_1} \Delta z_1(t, x) dx dt - \\ & - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left[\frac{\partial H_1(t, x, z_i(t, x), \bar{u}_1(t, x), \psi_1(t, x))}{\partial z_1} - \right. \\ & \left. - \frac{\partial H_1(t, x, z_1(t, x), u_1(t, x), \psi_1(t, x))}{\partial z_1} \right]' \Delta z_1(t, x) dx dt + \\ & + o_1(\|\Delta z_1(t_1, x_1)\|) + o_2(\|\Delta z_2(t_1, x_1)\|) + o_3(\|z_1(t_1, x_1)\|) - \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_4(t; \|z_1(t, x)\|) dx dt - \\ & - \int_{t_0}^{t_1} \int_{x_0}^{x_1} o_5(\|z_1(t, x)\|) dx dt - \int_{t_1}^{t_2} \int_{x_0}^{x_1} o_6(\|z_1(t, x)\|) dx dt. \quad (31) \end{aligned}$$

Suppose that $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ is an arbitrary Lebesgue point of the control $u_1(t, x)$, $v_1 \in U_1$ is an arbitrary vector, and $\varepsilon > 0$ is an arbitrary sufficiently small number such that $\theta + \varepsilon < t_1$ and $\xi + \varepsilon < x_1$.

The special increment of the control function $u_1(t, x)$ will be determined from the formula

$$\Delta u_1(t, x; \varepsilon) = \begin{cases} v_1 - u_1(t, x), & (t, x) \in [\theta, \theta + \varepsilon) \times [\xi, \xi + \varepsilon), \\ 0, & (t, x) \in D_1 \setminus [\theta, \theta + \varepsilon) \times [\xi, \xi + \varepsilon). \end{cases} \quad (32)$$

Given this formula and estimates (26), (29) from increment formula (31) we get the validity of the expansion

$$\begin{aligned} & J(u_1(t, x) + \Delta u_1(t, x; \varepsilon), u_2(t, x)) - J(u_1(t, x), u_2(t, x)) = \\ & = -\varepsilon^2 [H_1(\theta, \xi, z_1(\theta, \xi), v_1, \psi_1(\theta, \xi)) - H_1(\theta, \xi, z_1(\theta, \xi), u_1(\theta, \xi), \psi_1(\theta, \xi))] + o(\varepsilon^2). \quad (33) \end{aligned}$$

Suppose now that $\Delta u_1(t, x) = 0$, and the special increment of the control function $u_2(t, x)$ will be determined from the formula

$$\Delta u_2(t, x; \varepsilon) = \begin{cases} v_2 - u_2(t, x), & (t, x) \in [\theta, \theta + \mu) \times [\xi, \xi + \mu), \\ 0, & (t, x) \in D_2 \setminus [\theta, \theta + \mu) \times [\xi, \xi + \mu). \end{cases} \quad (34)$$

Here, $v_2 \in U_2$ is an arbitrary vector, $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ is an arbitrary Lebesgue point of the control $u_2(t, x)$, and $\mu > 0$ is an arbitrary sufficiently small number such that $\theta + \mu < t_2$ and $\xi + \mu < x_1$.

Given estimate (30) and formula (34) from increment formula (24) of the quality functional, we will have

$$J(u_1(t, x), u_2(t, x) + \Delta u_2(t, x; \mu)) - J(u_1(t, x), u_2(t, x)) =$$

$$= -\mu^2 [H_1(\theta, \xi, z_2(\theta, \xi), v_2, \psi_2(\theta, \xi)) - H_2(\theta, \xi, z_2(\theta, \xi), u_2(\theta, \xi), \psi_2(\theta, \xi))] + o(\mu^2). \quad (35)$$

From expansions (33) and (35) it follows that the following statement is true.

Theorem. The optimality of the admissible control $(u_1(t, x), u_2(t, x))$ requires that the condition of the maximum

$$\max_{v_1 \in U_1} H_1(\theta, \xi, z_1(\theta, \xi), v_1, \psi_1(\theta, \xi)) = H_1(\theta, \xi, z_1(\theta, \xi), u_1(\theta, \xi), \psi_1(\theta, \xi)),$$

$$\max_{v_2 \in U_2} H_2(\theta, \xi, z_2(\theta, \xi), v_2, \psi_2(\theta, \xi)) = H_2(\theta, \xi, z_2(\theta, \xi), u_2(\theta, \xi), \psi_2(\theta, \xi))$$

should be true for all $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ and $(\theta, \xi) \in [t_1, t_2) \times [x_0, x_1)$, respectively.

The proved theorem is an analogue of the Pontryagin maximum principle for the problem under investigation.

5. Conclusion

This study considers one problem of optimal control of a variable structure with distributed parameters described by a set of hyperbolic integro-differential equations and Volterra integral equations.

Applying one variant of the method of increments, an analogue of the Pontryagin maximum principle is proved.

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