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An analogue of the linearized maximum principle and Euler's equation in an optimal control problem for a discrete Rosser system

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ABSTRACT

We consider a discrete two-parameter optimal control problem described by a system of Rosser equations. Under different smoothness conditions on the parameters of the problem, a number of first-order necessary optimality conditions are obtained.

1. Introduction

In [1] a linear problem of optimal control for a Rosser system [2-6] with a linear quality functional was studied.

In this study we investigate a nonlinear problem of optimal control of a Rosser system with a nonlinear quality criterion.

Analogues of the linearized maximum condition and Euler's equation are proved (see, e.g., [7, 8]).

2. Problem statement

Suppose that the controlled process is described by the following system of two-dimensional difference equations

$$\begin{aligned} &z(t+1,x)=f\big(t,x,z(t,x),y(t,x),u(t,x)\big),\\ &t\in\{t_0,t_0+1,\dots,t_1-1\}, x\in\{x_0,x_0+1,\dots,x_1\},\\ &y(t,x+1)=g\big(t,x,z(t,x),y(t,x),u(t,x)\big), \end{aligned} \tag{1}$$

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$$t \in \{t_0, t_0+1, \dots, t_1\}, x \in \{x_0, x_0+1, \dots, x_1-1\}, \tag{2}$$

with the boundary conditions

$$z(t_0, x) = a(x), \qquad x \in \{x_0, x_0+1, \dots, x_1\}, \tag{3}$$

$$y(t, x_0) = b(t), \quad t \in \{t_0, t_0+1, \dots, t_1\}, \tag{4}$$

where, $f(t,x,z,y,u)$ and $g(t,x,z,y,u)$ are specified n-dimensional vector functions discrete in (t,x) and continuous in (z,y,u) with partial derivatives with respect to (z,y,u), $a(x)$ and $b(t)$ are specified n-dimensional discrete vector functions, t_0, t_1, x_0, x_1 are specified $numbers$, with the differences $t_1 – t_0$ and $x_1 - x_0$ being natural numbers, and $u(t,x)$ is a r-dimensional, discrete vector control actions with values from the specified non-empty, bounded and convex set $U \subset R^r$, i.e.,

$$u(t,x) \in U \subset R^r, t = t_0, t_0+1, \dots, t_1-1, x = x_0, x_0+1, \dots, x_1-1. \tag{5}$$

Each control function with the above properties will be called an admissible control.

Suppose that $\varphi_1(x,z)$ and $\varphi_2(t,y)$ are specified scalar functions discrete in t and x and continuously differentiable in z and y, respectively.

On the solutions of boundary problem (1)-(4) generated by all possible admissible controls we determine the functional

$$S(u) = \sum_{x=x_0}^{x_1-1} \varphi_1(x, z(t_1, x)) + \sum_{t=t_0}^{t_1-1} \varphi_2(t, y(t, x_1)). \tag{6}$$

It is required to minimize quality functional (6) with constraints (1)-(5).

The admissible control $u(t,x)$, that affords the minimum to functional (6) with constraints (1)-(5) will be called an optimal control, and the corresponding process $(u(t,x), z(t,x), y(t,x))$ – an optimal process.

3. Special increments of the quality criterion and necessary optimality conditions

Suppose that $(u(t,x), z(t,x), y(t,x))$ and $\big(\bar{u}(t,x) = u(t,x) + \Delta u(t,x), \bar{z}(t,x) = z(t,x) + \Delta z(t,x), \bar{y}(t,x) = y(t,x) + \Delta y(t,x)\big)$ are some admissible processes.

We will write the increment of functional (6) corresponding to these admissible processes as

$$\Delta S(u) = S(\bar{u}) - S(u) = \sum_{x=x_0}^{x_1-1} \Big(\varphi_1\big(x, \bar{z}(t_1,x)\big) - \varphi_1\big(x, z(t_1,x)\big)\Big) + $$
$$+ \sum_{t=t_0}^{t_1-1} \big(\varphi_2(t, \bar{y}(t,x_1)) - \varphi_2(t, y(t,x_1))\big). \tag{7}$$

On the other hand, it is evident that the increment $\big(\Delta z(t,x), \Delta y(t,x)\big)$ of the state $\big(z(t,x), y(t,x)\big)$ will be the solution to the problem

$$\Delta z(t+1,x) = f\big(t,x,\bar{z}(t,x),\bar{y}(t,x),\bar{u}(t,x)\big) - f\big(t,x,z(t,x),y(t,x),u(t,x)\big), \tag{8}$$

$$\Delta z(t_0, x) = 0, \tag{9}$$

$$\Delta y(t,x+1) = g\big(t,x,\bar{z}(t,x),\bar{y}(t,x),\bar{u}(t,x)\big) - g\big(t,x,z(t,x),y(t,x),u(t,x)\big) \tag{10}$$

$$\Delta y(t, x_0) = 0. \tag{11}$$

Suppose that $\psi(t,x)$ and $p(t,x)$ are as yet arbitrary n- dimensional vector functions.

From relations (8) and (10) we have that

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t,x)\Delta z(t+1,x)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}p'(t,x)\Delta y(t,x+1)=$$

$$=\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\Big[\psi'(t,x)\left(f\big(t,x,\bar z(t,x),\bar y(t,x),\bar u(t,x)\big)-f\big(t,x,z(t,x),y(t,x),u(t,x)\big)\right)+$$

$$+p'(t,x)\Big(g\big(t,x,\bar z(t,x),\bar y(t,x),\bar u(t,x)\big)-g\big(t,x,z(t,x),y(t,x),u(t,x)\big)\Big)\Big]. \quad (12)$$

Taking into account boundary conditions (9) and (11), it is easy to prove that

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t,x)\Delta z(t+1,x)=$$

$$=\sum_{x=x_0}^{x_1-1}\psi'(t_1-1,x)\Delta z(t_1,x)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t-1,x)\Delta z(t,x), \quad (13)$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}p'(t,x)\Delta y(t,x+1)=$$

$$=\sum_{t=t_0}^{t_1-1}p'(t,x_1-1)\Delta y(t,x_1)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}p'(t,x-1)\Delta y(t,x). \quad (14)$$

We will introduce an analogue of the Hamilton-Pontryagin function for the problem under investigation in the following form:

$$H\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)=\psi'(t,x)f\big(t,x,z(t,x),y(t,x),u(t,x)\big)+$$

$$+p'(t,x)\,g\big(t,x,z(t,x),y(t,x),u(t,x)\big)\Big).$$

Taking into account identities (12)-(14), increment (7) of quality functional (6) is given in the form

$$\Delta S(u)=\sum_{x=x_0}^{x_1-1}\Big(\varphi_1\big(x,\bar z(t_1,x)\big)-\varphi_1\big(x,z(t_1,x)\big)\Big)+\sum_{t=t_0}^{t_1-1}\Big(\varphi_2\big(t,\bar y(t,x_1)\big)-\varphi_2\big(t,y(t,x_1)\big)\Big)+$$

$$+\sum_{x=x_0}^{x_1-1}\psi'(t_1-1,x)\Delta z(t_1,x)+\sum_{t=t_0}^{t_1-1}p'(t,x_1-1)\Delta y(t,x_1)+$$

$$+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t-1,x)\Delta z(t,x)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}p'(t,x-1)\Delta y(t,x)-$$

$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\Big(H\Big(t,x,\bar z(t,x),\bar y(t,x),\bar u(t,x),\bar\psi(t,x),\bar p(t,x)\Big)-$$

$$-H\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)\Big). \quad (15)$$

By virtue of the smoothness conditions imposed on the parameters of the problem under investigation, we have from increment formula (15) that

$$\Delta S(u)=\sum_{x=x_0}^{x_1-1}\frac{\partial\varphi_1'\big(x,z(t_1,x)\big)}{\partial z}\Delta z(t_1,x)+\sum_{x=x_0}^{x_1-1}o_1(\|\Delta z(t_1,x)\|)+$$
$$+\sum_{t=t_0}^{t_1-1}\frac{\partial\varphi_2'\big(t,y(t,x_1)\big)}{\partial y}\Delta y(t,x_1)+\sum_{t=t_0}^{t_1-1}o_2(\|\Delta y(t,x_1)\|)+$$
$$+\sum_{x=x_0}^{x_1-1}\psi'(t_1-1,x)\Delta z(t_1,x)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t-1,x)\Delta z(t,x)+$$
$$+\sum_{t=t_0}^{t_1-1}p'(t,x_1-1)\Delta y(t,x_1)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}p'(t,x-1)\Delta y(t,x)-$$
$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\Bigg[\frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u}\Delta u(t,x)+$$
$$-\frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial z}\Delta z(t,x)+$$
$$+\frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial y}\Delta y(t,x)$$
$$+o_3(\|\Delta z(t,x)\|+\|\Delta y(t,x)\|+\|\Delta u(t,x)\|)\Bigg]. \tag{16}$$

Here $\|\alpha\|$ is the norm of the vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)'$ determined by the formula $\|\alpha\|=\sum_{i=1}^{n}|\alpha_i|$, and $o(\alpha)$ is a quantity of a higher order than α, i.e., $o(\alpha)/\alpha\to 0$ at $\alpha\to 0$, and the dash ($'$) is a transpose operation.

Suppose that the vector of the function $\psi(t,x)$ and $p(t,x)$ are solutions of the following linear difference problems

$$\psi(t-1,x)=\frac{\partial H\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial z}, \tag{17}$$

$$\psi(t_1-1,x)=-\frac{\partial\varphi_1\big(x,z(t_1,x)\big)}{\partial z}, \tag{18}$$

$$p(t,x-1)=\frac{\partial H\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial y}, \tag{19}$$

$$p(t,x_1-1)=-\frac{\partial\varphi_2\big(t,y(t,x_1)\big)}{\partial y}. \tag{20}$$

Problems (17)-(18) and (19)-(20) will be called an adjoint system (see, e.g., [7, 8]) for the problem under investigation.

If relations (17)-(20) are satisfied, increment formula (16) takes the following form

$$\Delta S(u)=-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u}\Delta u(t,x)+$$

$$+\sum_{x=x_0}^{x_1-1}o_1(\|\Delta z(t_1,x)\|)+\sum_{t=t_0}^{t_1-1}o_2(\|\Delta y(t,x_1)\|)-$$

$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} o_3(\|\Delta z(t,x)\|+\|\Delta y(t,x)\|+\|\Delta u(t,x)\|). \tag{21}$$

Further on we will need estimates of the norm of increment of the states $z(t,x)$ and $y(t,x)$. Taking into account boundary conditions (9) and (11), we can write that

$$\Delta z(t+1,x)=\sum_{\tau=t_0}^{t}\big[f(\tau,x,\bar{z}(\tau,x),\bar{y}(\tau,x),\bar{u}(\tau,x))-$$
$$-f(\tau,x,z(\tau,x),y(\tau,x),u(\tau,x))-\Delta z(\tau,x)\big], \tag{22}$$
$$\Delta y(t,x+1)=\sum_{s=x_0}^{x}\big[g(t,s,\bar{z}(t,s),\bar{y}(t,s),\bar{u}(t,s))-$$
$$-g(t,s,z(t,s),y(t,s),u(t,s))-\Delta y(t,s)]. \tag{23}$$

Therefore, passing to the norm in identities (22) and (23) and using the Lipschitz condition and the triangle rule in the norm we have that

$$\|\Delta z(t+1,x)\|\le L_1\sum_{\tau=t_0}^{t}[\|\Delta z(\tau,x)\|+\|\Delta y(\tau,x)\|+\|\Delta u(\tau,x)\|], \tag{24}$$

$$\|\Delta y(t,x+1)\|\le L_2\sum_{s=x_0}^{x}[\|\Delta z(t,s)\|+\|\Delta y(t,s)\|+\|\Delta u(t,s)\|], \tag{25}$$

where $L_i=const>0, i=1,2$ are some constants (Lipschitz constants).

Successively applying to inequalities (24) and (25) the discrete analogue of Gronwall's formula (see, e.g., [8]), we have that

$$\|\Delta z(t,x)\|\le L_3\sum_{\tau=t_0}^{t}[\|\Delta y(\tau,x)\|+\|\Delta u(\tau,x)\|], \tag{26}$$

$$\|\Delta y(t,x)\|\le L_4\sum_{s=x_0}^{x}[\|\Delta z(t,s)\|+\|\Delta u(t,s)\|], \tag{27}$$

where $L_3>0, L_4>0$ are some constants.

Given inequalities (26) and (27) with each other, we have

$$\|\Delta z(t,x)\|\le L_5\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}[\|\Delta z(\tau,s)\|+\|\Delta u(\tau,s)\|]+L_3\sum_{\tau=t_0}^{t}\|\Delta u(\tau,x)\|, \tag{28}$$

$$\|\Delta y(t,x)\|\le L_6\sum_{s=x_0}^{x}\sum_{\tau=t_0}^{t}[\|\Delta y(\tau,s)\|+\|\Delta u(\tau,s)\|]+L_4\sum_{s=x_0}^{x}\|\Delta u(t,s)\|, \tag{29}$$

where $L_5>0, L_6>0$ are some constants.

From inequalities (28) and (29), applying the two-dimensional discrete analogue of the Gronwall-Bellman lemma (see, e.g., [9, 10]), we arrive at the estimates

$$\|\Delta z(t,x)\|\le L_7\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\|\Delta u(\tau,s)\|+\sum_{\tau=t_0}^{t}\|\Delta u(\tau,x)\|\right], \tag{30}$$

$$\|\Delta y(t,x)\| \le L_8 \left[\sum_{s=x_0}^{x} \sum_{\tau=t_0}^{t} \|\Delta u(\tau,s)\| + \sum_{s=x_0}^{x} \|\Delta u(t,s)\| \right], \tag{31}$$

where $L_7 > 0, L_8 > 0$ are some constants.

We now proceed to the calculation of the special functional increment.

Suppose that $\varepsilon \in [0,1]$ is an arbitrary number, and $v(t,x)$ is an arbitrary admissible control. The special increment of the admissible control $u(t,x)$ will be determined from the formula

$$\Delta u_\varepsilon(t,x) = \varepsilon[v(t,x) - u(t,x)]. \tag{32}$$

We denote by $(\Delta z_\varepsilon(t,x), \Delta y_\varepsilon(t,x))$ the special increment of the state $(z(t,x), y(t,x))$ corresponding to increment (32) of the control $u(t,x)$.

From estimates (30) (31) it follows that

$$\|\Delta z_\varepsilon(t,x)\| \le \varepsilon L_7 \left[\sum_{\tau=t_0}^{t} \sum_{s=x_0}^{x} \|v(\tau,s) - u(\tau,s)\| + \sum_{\tau=t_0}^{t} \|v(\tau,x) - u(\tau,x)\| \right], \tag{33}$$

$$\|\Delta y_\varepsilon(t,x)\| \le \varepsilon L_8 \left[\sum_{s=x_0}^{x} \sum_{\tau=t_0}^{t} \|v(\tau,s) - u(\tau,s)\| + \sum_{s=x_0}^{x} \|v(t,s) - u(t,s)\| \right]. \tag{34}$$

Taking into account formula (32) and estimates (33), (34) from increment formula (21) of functional (6) we get that

$$S(u + \Delta u_\varepsilon) - S(u) =$$
$$= -\varepsilon \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} \frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u} \times$$
$$\times [v(t,x) - u(t,x)] + o(\varepsilon). \tag{35}$$

From expansion (35) follows

Theorem 1. It is necessary for the optimality of the admissible control $u(t,x)$ that the inequality

$$\sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} \frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u} [v(t,x) - u(t,x)] \le 0 \tag{36}$$

hold for all admissible controls $v(t,x)$.

Inequality (36) is an analogue of the linearized maximum condition [7, 8].

Note that from it we can obtain a pointwise linearized necessary optimality condition.

Suppose now that the set U is open. Then the special increment of the admissible control $u(t,x)$ can be determined from the formula

$$\Delta u_\mu(t,x) = \mu \delta\, u(t,x). \tag{37}$$

Here, μ is a sufficiently small number in absolute value, and $\delta u(t,x) \in R^r, t = t_0, t_0 + 1, \dots, t_1 - 1, x = x_0, x_0 + 1, \dots, x_1 - 1$ is an arbitrary discrete and bounded vector function (admissible control variation).

From estimates (30) and (31) it immediately follows that

$$\|\Delta z(t,x)\| \le |\mu| L_7 \left[\sum_{\tau=t_0}^{t} \sum_{s=x_0}^{x} \|\delta u(\tau,s)\| + \sum_{\tau=t_0}^{t} \|\delta u(\tau,x)\| \right], \tag{38}$$

$$\|\Delta y(t,x)\| \le \|\mu\| L_8 \left[\sum_{s=x_0}^{x}\sum_{\tau=t_0}^{t}\|\delta u(\tau,s)\| + \sum_{s=x_0}^{x}\|\delta u(t,s)\|\right]. \tag{39}$$

Taking into account formula (37) and estimates (38), (39) from increment formula (21) of functional (6) we get that

$$S(u+\mu\delta u) - S(u) =$$

$$= -\mu \sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} \frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u}\delta u(t,x) + o(\mu).$$

Therefore, the first variation (in the classical sense) of functional (6) has the form

$$\delta^1 S(u{:}\,\delta u) = -\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} \frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u}\delta u(t,x). \tag{40}$$

As is known (see, e.g., [7, 8]), along the optimal control the first variation of the quality functional is zero. Therefore, it follows from (40) that along the optimal control $u(t,x)$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} \frac{\partial H'\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u}\delta u(t,x) = 0.$$

Hence, by virtue of arbitrariness of $\delta u(t,x)$ follows

Theorem 2. If the set U is open, then it is necessary for the optimality of the admissible control $u(t,x)$ that the relation

$$\frac{\partial H|\big(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),p(t,x)\big)}{\partial u} = 0 \tag{41}$$

hold for all $= t_0, t_0+1, \dots, t_1-1, x = x_0, x_0+1, \dots, x_1-1$.

Relation (41) is an analogue of the classical Euler equation (see, e.g., [7]) for the problem under investigation.

4. Conclusion

The article studies a class of discrete two-parameter optimal control problems described by a discrete analogue of the canonical first-order hyperbolic equation (Rosser model).

An analogue of the linearized maximum condition is proved for the assumption of convexity of the control domain. In the case of openness of the control domain, the first variation of the quality functional is calculated and the analog of Euler's equation is obtained from the condition of the first variation of the quality criterion along the optimal control being equal to zero.

A necessary optimality condition in the form of an analogue of Euler's equation is established.

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